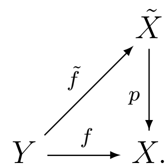
21. The homotopy lifting property

Definition 21.1. *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space. If $f\colon Y \longrightarrow X$ is a continuous function then a* ***lift of*** *f* ***along*** *p is a continuous function $\tilde{f}\colon X \longrightarrow \tilde{Y}$ such that the following diagram commutes*



Note that the diagram commutes if and only if

$$f = p \circ \tilde{f}.$$

Lemma 21.2 (Uniqueness of lifts)**.** *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space and let $f\colon Y \longrightarrow X$ be a continuous function.*

If $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of f then the set

$$E = \{\, y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \,\}$$

is both open and closed in Y.

In particular, if Y is connected then either $E = \varnothing$ or $E = Y$.

Proof. We first show that E is open. Suppose that $y \in E$. Pick an open neighbourhood $U \subset X$ of $f(x)$ that is evenly covered by p.

Then

$$p^{-1}(U) = \coprod_{\alpha \in \Lambda} V_\alpha.$$

As $\tilde{f}_1(y) = \tilde{f}_2(y)$ it follows that $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ both belong to the same V_β. Let

$$N = \tilde{f}_1^{-1}(V_\beta) \cap \tilde{f}_2^{-1}(V_\beta).$$

Then N is an open subset of Y. We have

$$p|_{V_\beta} \circ \tilde{f}_1|_N = f|_N = p|_{V_\beta} \circ \tilde{f}_2|_N.$$

As $p|_{V_\beta}$ is a homeomorphism, it is certainly a bijection and it follows that

$$\tilde{f}_1|_N = \tilde{f}_2|_N.$$

Thus $y \in N \subset E$ and so E is open.

Now we show that E is closed. There are two ways to proceed.

For the first we will show the complement is open. Pick $y \notin E$. We proceed as above and we use similar notation. As $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ we conclude that $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ belong to $V_\beta \neq V_\gamma$. But then

$$y \in N = \tilde{f}_1^{-1}(V_\beta) \cap \tilde{f}_2^{-1}(V_\gamma)$$

is an open subset of the complement of E. Thus the complement is open and E is closed.

For the second way, assume that X is Hausdorff. It follows easily that $\tilde{X}$ is Hausdorff. But then the locus where two functions to a Hausdorff space are equal is always closed. □

Theorem 21.3 (Homotopy lifting property)**.** *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space and let $F\colon Y \times I \longrightarrow X$ be a homotopy from f_0 to f_1. Suppose that there is a lift $\tilde{f}_0$ of f_0. Then we may lift F to a homotopy making the following diagram commute*

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}_0} & \tilde{X} \\ {\scriptstyle i}\big\downarrow & \overset{\tilde{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow{F} & X, \end{array}$$

where $i\colon Y \longrightarrow Y \times I$ is the continuous function $i(y) = (y, 0)$.

Proof. Pick an open cover of X

$$\mathcal{U} = \{ U_\alpha \,|\, \alpha \in \Lambda \}$$

by open sets U_α that are evenly covered. We may write

$$p^{-1}(U_\alpha) = \coprod_{\beta \in \Lambda_\alpha} V_\beta$$

where each V_β is homeomorphic to U_α.

We get an open cover of $Y \times I$

$$\mathcal{V} = \{ F^{-1}(U_\alpha) \,|\, \alpha \in \Lambda \}.$$

If we fix $y_0 \in Y$ then we get an open cover of $\{ y_0 \} \times I$. Pick $N = N(y_0)$ so that

$$\frac{1}{N} < \delta,$$

the Lebesgue number of the cover. Then each path

$$F|_{\{ y_0 \} \times [i/N,(i+1)/N]} \colon \{ y_0 \} \times [i/N, (i+1)/N] \longrightarrow X$$

lands in (at least) one U_α. But then the Tube Lemma implies that we may find an open neighbourhood W_{y_0} of y_0 such that

$$W_{y_0} \times [i/N, (i+1)/N]$$

lands entirely in U_α, for each i.

We now lift $F|_{W_{y_0} \times [i/N,(i+1)/N]}$ to $\tilde{F}|_{W_{y_0} \times [i/N,(i+1)/N]}$, for each i. We start with $i = 0$. By assumption we land in some U_α. We already know how to lift $F|_{W_{y_0} \times \{ 0 \}}$ to $\tilde{F}|_{W_{y_0} \times \{ 0 \}}$. Suppose this lands in V_β. Since

$$p|_{V_\beta} \colon V_\beta \longrightarrow U_\alpha$$

is a homeomorphism, we define

$$\tilde{F}|_{W_{y_0}\times[0,1/N]} = (p|_{V_\beta})^{-1} \circ F|_{W_{y_0}\times[0,1/N]}.$$

Now we keep repeating the same argument. We lift $F|_{W_{y_0}\times[1/N,2/N]}$ to $\tilde{F}|_{W_{y_0}\times[1/N,2/N]}$, then we lift $F|_{W_{y_0}\times[2/N,3/N]}$ to $\tilde{F}|_{W_{y_0}\times[2/N,3/N]}$, and so on.

Continuing in this way, after N steps, we construct a lift of $F|_{W_{y_0}\times I}$ to $\tilde{F}|_{W_{y_0}\times I}$.

The final thing to check is that we can patch these functions together. Suppose that

$$(y,t) \in (W_{y_0} \times I) \cap (W_{y_1} \times I).$$

In this case

$$y \in W_{y_0} \cap W_{y_1}.$$

As as our two lifts have to agree at

$$(y,0) \in (W_{y_0\times I}) \cap (W_{y_1\times I})$$

and I is connected (21.2) implies they have to agree on

$$\{\, y \,\} \times I.$$

But then they agree at

$$(y,t) \in \{\, y \,\} \times I. \qquad \square$$

Even the case when $Y = \{\, * \,\}$ is a point is interesting:

Corollary 21.4. *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space and let $\gamma\colon I \longrightarrow X$ be a path starting at x_0.*

If $\tilde{x}_0 \in \tilde{X}$ is a point such that $p(\tilde{x}_0) = x_0$ then there is a unique path $\tilde{\gamma}\colon I \longrightarrow \tilde{X}$ such that

(1) $\tilde{\gamma}(0) = \tilde{x}_0$, and
(2) $p \circ \tilde{\gamma} = \gamma$.

Note the importance of base points. If we don't fix $\tilde{x}_0$ then it is not true that there is a unique lift.

Corollary 21.5. *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space and let $\gamma_0\colon I \longrightarrow X$ and $\gamma_1\colon I \longrightarrow X$ be paths from x_0 to x_1. Let $\tilde{\gamma}_0\colon I \longrightarrow \tilde{X}$ and $\tilde{\gamma}_1\colon I \longrightarrow \tilde{X}$ be paths starting at $\tilde{x}_0$ lifting γ_0 and γ_1.*

If $\gamma_0 \sim \gamma_1$ as paths then $\tilde{\gamma}_0 \sim \tilde{\gamma}_1$ as paths. In particular the endpoint $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$ does not depend on the lift.

Proof. Pick a homotopy $F\colon I^2 \longrightarrow X$ from f_0 to f_1. By (21.3) we may lift this to a homotopy $\tilde{F}\colon I^2 \longrightarrow \tilde{X}$ from $\tilde{f}_0$.

Now the path

$$\tilde{\gamma}\colon I \longrightarrow \tilde{X} \qquad \text{given by} \qquad \tilde{\gamma}(t) = \tilde{F}(t, 1)$$

is a lift of γ_1 starting at $\tilde{x}_0$. Thus $\tilde{\gamma} = \tilde{\gamma}_1$, by (21.2).

The paths

$$I \longrightarrow X \qquad \text{given by} \qquad F(0, t) = x_0$$

and

$$I \longrightarrow X \qquad \text{given by} \qquad F(1, t) = x_1$$

are constant. It follows that their lifts via $\tilde{F}$ are constant. Thus

$$\tilde{x}_0 = \tilde{F}(0, t) \qquad \text{and} \qquad \tilde{x}_1 = \tilde{F}(i, t)$$

are also constant. Thus $\tilde{F}$ is a homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$ as paths from $\tilde{x}_0$ to $\tilde{x}_1$. □