8. Functors

A functor is a way to compare two categories:

Definition 8.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A* ***(covariant) functor*** *F from $\mathcal{C}$ to $\mathcal{D}$, associates to every object X of $\mathcal{C}$ an object $F(X)$ or $\mathcal{D}$ and to every morphism*

$$f\colon X \longrightarrow Y$$

in $\mathcal{C}$ a morphism

$$F(f)\colon F(X) \longrightarrow F(Y)$$

such that

(1) $F(id_X) = id_{F(X)}$, and
(2) If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two morphisms in $\mathcal{C}$ then

$$F(g \circ f) = F(g) \circ F(f)\colon F(X) \longrightarrow F(Z).$$

Contravariant functors are defined similarly to covariant functors, except two key differences. Firstly the map $F(f)$ goes the other way

$$F(f)\colon F(Y) \longrightarrow F(X).$$

Accordingly composition goes the other way

$$F(g \circ f) = F(f) \circ F(g)\colon F(Z) \longrightarrow F(X).$$

Given any category, there is always an identity functor, which is the identity on objects and morphisms.

If we start with the category of topological spaces, perhaps the simplest functor is to the category of sets. This functor sends a topological space to its underlying set and a continuous function to the underlying function. This is called a forgetful functor.

There are lots of forgetful functors one can write down and even forgetful functors can be surpisingly useful. For example, if we start with the category of groups, there is also a forgetful functor to the category of sets. It sends a group to the underlying set and a group homomorphism to the underlying function.

It we start with the category of simplicial complexes there are natural functors C_m to the category of abelian groups. To a simplicial complex $\mathcal{K}$ one associates the group of m-chains $C_m(\mathcal{K})$ and to a simplicial map $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ the pushforward map

$$C_m(f) = f_*\colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L})$$

It is clear that the identity goes to the identity and we already checked that

$$C_m(g \circ f) = C_m(g) \circ C_m(f).$$

The other obvious such functor is H_m. To a simplicial complex $\mathcal{K}$ one associates the mth homomorphism group $H_m(\mathcal{K})$ and to a simplicial map $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ the pushforward map

$$H_m(f) = f_*\colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L})$$

It is clear that the identity goes to the identity and we already checked that

$$H_m(g \circ f) = H_m(g) \circ H_m(f).$$